RELATIVE CATEGORIES: ANOTHER MODEL FOR THE HOMOTOPY THEORY OF HOMOTOPY THEORIES

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Abstract. We lift Charles Rezk's complete Segal space model structure on the category of simplicial spaces to a Quillen equivalent one on the category of relative categories.

Contents

1. Introduction

1.1. **Summary.** The usefulness of homotopical and (co-)homological methods in so many parts of modern mathematics seems to be due to the following two facts:

(i) One often runs into what we will call *relative categories*, i.e. pairs $(\boldsymbol{C}, \boldsymbol{W})$ consisting of a category $\boldsymbol{C}$ and a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ which contains all the objects of $\boldsymbol{C}$ and of which the maps are called *weak equivalences* because one would have liked them to behave like isomorphisms.

(ii) Such a relative category $(\boldsymbol{C}, \boldsymbol{W})$ is in essence a *homotopy theory* because one can not only form the *localization* of $\boldsymbol{C}$ with respect to $\boldsymbol{W}$ (often called its *homotopy category*) which is the category obtained from $\boldsymbol{C}$ by "formally inverting" all the weak equivalences, but one can also form the more delicate *simplicial localization* of $\boldsymbol{C}$ with respect to $\boldsymbol{W}$, which is a *simplicial category* (i.e, a category enriched over simplicial sets) with the same objects as $\boldsymbol{C}$.

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In this paper we are interested in the fact that two such relative categories give rise to the "same" homotopy theory if they can be connected by a finite zigzag of DK-*equivalences*, i.e. weak equivalences preserving functors which induce

— an *equivalence of categories* between their homotopy categories, and
— *weak (homotopy) equivalences* between the simplicial sets involved in their simplicial localizations.

One thus can ask

(i) whether there exists on the category **RelCat** of small relative categories and weak equivalence-preserving functors a model structure that is a *homotopy theory of homotopy theories* in the sense that it is Quillen equivalent to the ones considered by Julie Bergner, André Joyal, Charles Rezk, and others, and
(ii) whether the weak equivalences in this model structure are the DK equivalences.

Our main result in this paper is an *affirmative answer* to the first of these.

An affirmative answer to the second of these questions requires a better understanding of *simplicial localization functors* [BK1] and will be given in [BK2].

1.2. **Further details.** Our main result consists of proving that there exists a model structure on the category **RelCat** of (small) relative categories and weak equivalence preserving functors between them that is Quillen equivalent to Charles Rezk's complete Segal structure on the category s$\boldsymbol{S}$ of simplicial spaces (i.e. bisimplicial sets) and thus *is a model for the theory of* ∞*-categories* (or more precisely, $(\infty, 1)$-categories). We do this by showing that the Reedy model structure on s$\boldsymbol{S}$ and all its left Bousfield localizations (and hence in particular the just mentioned complete Segal structure) can be lifted to Quillen equivalent model structure on **RelCat**.

We also obtain for each such model structure on s$\boldsymbol{S}$ also a *conjugate* model structure on **RelCat** with the same weak equivalences and hence the same underlying relative category as the model structure discussed above. Moreover the *involution* of **RelCat** that sends each (small) relative category to its *opposite* is a Quillen equivalence (in fact an isomorphism) between these two model structures on **RelCat** and models the contractible space of nontrivial auto-equivalences of theories of $(\infty, 1)$-categories.

The proof is basically a relative version of Bob Thomason's arguments that the usual model structure on the category of simplicial sets can be lifted to a Quillen equivalent model structure on the category of (small) categories, combined with some ideas contained in a paper he wrote together with Dana Latch and Steve Wilson.

2. An overview

This paper consists essentially of three parts. The first part contains a

2.1. **Formulation of our main result.** This will be done in the first four sections, §§3–6.

(i) In §3 we introduce the category **RelCat** of (small) *relative categories* and *relative functors* between them and introduce in this category notions of *homotopic maps* and *homotopy equivalences*.

Moreover we introduce, following Thomason, a notion of *Dwyer maps* which are a kind of neighborhood deformation retracts with such cofibration-like properties (which we will verify in §9 (9.1–9.3) as being closed under retracts, pushouts, and (possibly transfinite) compositions.

(ii) In §4 we consider the special case of *relative posets* and define for them two kinds of *subdivisions*, a *terminal* one and an *initial* one which we will denote by ξ_t and ξ_i respectively. Unlike what happens in the case of (ordinary) posets, these two subdivisions are in general not each others' opposites, but only each others' *conjugates*. While Thomason needed only the iteration of one of them we will, for reasons which will become clear in §9 (9.4–9.6), need the *composition* $\xi = \xi_t\xi_i$ of the two of them, which we will refer to as the *two-fold subdivision*. Of course we could just as well have used the conjugate two-fold subdivision $\overline{\xi} = \xi_i\xi_t$. In that case, the *opposites* of our arguments then yield a Quillen equivalent *conjugate model structure* with the same weak equivalences, in which the cofibrations and fibrations are the opposites of ours.

(iii) In §5 we develop some preliminaries needed in order to formulate our main result.
   (a) We recall what is precisely meant by *lifting* a cofibrantly generated model structure.
   (b) We describe the *Reedy model structure* on the category $\mathrm{s}\boldsymbol{S}$ of bisimplicial sets, as well as its *left Bousfield localizations.*
   (c) We define two *adjunctions*

$$K_\xi\colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N_\xi \qquad \text{and} \qquad K\colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N$$

   of which the *first* is the adjunction which will allow us to *lift* the above ((iii)b) model structures on $\mathrm{s}\boldsymbol{S}$ to Quillen equivalent ones on **RelCat**.
   (d) We also formulate a *key lemma*, which states that the two right adjoints

$$N_\xi, N\colon \mathbf{RelCat} \longrightarrow \mathrm{s}\boldsymbol{S}$$

   are naturally Reedy equivalent. At a crucial point (in §10) in the proof of our main result, this key lemma enables us to use, instead of the functor N_ξ, the much simpler *simplicial nerve* functor N of Charles Rezk [R] (who called it the *classifying diagram* functor).

(iv) In §6 we state our main results and mention some of its consequences.
   (a) Our main result consists of the lifts mentioned above and hence in particular the lifts of Rezk's complete Segal model structure on $\mathrm{s}\boldsymbol{S}$ to a Quillen equivalent one on **RelCat**.
   (b) Moreover, we note that for each of the resulting model structures on **RelCat**, there is a *conjugate* model structure that is connected to it by the *involution* of **RelCat** (1.2, I).
   (c) We also note that the two model structures on **RelCat** lifted from Rezk's complete Segal structure on $\mathrm{s}\boldsymbol{S}$ are each models for the theory of $(\infty,1)$-*categories*, and that the involution relating them models the *contractible space of nontrivial auto-equivalences of the theory of* $(\infty,1)$-*categories.*

(d) Finally, we observe, after reformulating Thomason's result in our language, that our Quillen equivalences (iv)a and Thomason's Quillen equivalences are tightly connected by a simple pair of Quillen pairs.

The second part of the paper consists of

2.2. **A proof of the above key lemma mentioned above.** This will be dealt with in §7 and §8.

Thomason proved this lemma for simplicial sets by using the fact that for every simplicial set Y, the natural map $Y \to \mathrm{Ex}\, Y$ [K] is a weak equivalence. However, as we were not able to relativize this result, we will instead relativize a quite different argument that is contained in a paper that he wrote jointly with Dana Latch and Steve Wilson [LTW].

In §7 we do the following:

(i) We note that the category **RelCat** is *closed monoidal* and that the homotopy relation in **RelCat** is compatible with this closure.
(ii) We prove that, on *finite* relative posets, the subdivision functor ξ_t, ξ_i and ξ are homotopy preserving.
(iii) We describe sufficient conditions on functors $\mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$ in order that they send homotopic maps in **RelCat** to homotopic maps in $\mathrm{s}\boldsymbol{S}$.

Finally, in §8,

(iv) we use these results to relativize the arguments used in the paper [LTW].

The third and last part of the paper consists of

2.3. **A proof of the main result.** This will be done in §9 and §10.

The first of these, §9, is devoted to Dwyer maps.

(i) In 9.1–9.3 we show that Dwyer maps are closed under *retracts*, *pushouts* and (possibly transfinite) *compositions*.
(ii) In 9.4 we describe *sufficient conditions* on a relative inclusion of relative posets in order that its terminal subdivision is a Dwyer map and in 9.5 we use this to show that if, for every pair of integers $p, q \geq 0$, $\Delta[p,q]$ and $\partial\Delta[p,q]$ respectively denote the standard (p,q)-bisimplex and its boundary, then the inclusion $\partial\Delta[p,q] \to \Delta[p,q]$ induces a relative inclusion (2.1(iii))
$$K_\xi \partial\Delta[p,q] \longrightarrow K_\xi \Delta[p,q] \in \mathbf{RelPos}$$
which is a Dwyer map.
(iii) In 9.6 we then use (i) and (ii) to show that every *monomorphism* $L \to M \in \mathrm{s}\boldsymbol{S}$ gives rise to a *Dwyer map* $K_\xi L \to K_\xi M \in \mathbf{RelCat}$.

We finally complete the proof of our main result in §10.

(iv) In 10.1 and 2 we relativize another key lemma of Thomason by showing that, *up to a weak equivalence* in the Reedy model structure on $\mathrm{s}\boldsymbol{S}$ *pushouts along a Dwyer map commute with the simplicial nerve functor* N, and
(v) note that, in view of the first key lemma (2.1(iii)) *the same holds for the functor* N_ξ.
(vi) In 10.3 and 4 we deduce from this that *the unit* $1 \to N_\xi K_\xi$ *of the adjunction*
$$K_\xi \colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N_\xi$$
is a natural Reedy equivalence and that *a map* $L \to M \in \mathrm{s}\boldsymbol{S}$ *is a Reedy equivalence iff the induced map* $N_\xi K_\xi L \to N_\xi K_\xi M \in \mathrm{s}\boldsymbol{S}$ *is so.*

(vii) In 10.5 we then combine these results with the ones of §9 to finally prove our main result.

We end with a

2.4. **Remark.** The reader may wonder why we (and Thomason) did not prove 2.3 directly, i.e. without using the simplicial nerve functor N, as this would have avoided the need for the first key lemma (2.1(iii)d). The reason is that such a proof would probably have been much more complicated than the present approach, as the proof of 2.3(iv) relies heavily on the fact that the relative posets involved in the definition of the functor N all have an *initial object*, something that is not at all the case for the functor N_ξ.

3. Relative categories

In this section we

(i) introduce the category **RelCat** of (small) *relative categories* and *relative functors* between them,
(ii) define a *homotopy relation* on **RelCat**, and
(iii) use this to describe a very useful class of relative functors which are a kind of *neighborhood deformation retracts* and have *cofibration-like* properties and which, following Thomason [T1], we will call *Dwyer maps.*

3.1. **Relative categories and functors.** A **relative category** will be a pair $\boldsymbol{C}$ consisting of

(i) a category, called the **underlying category** and denoted by $\operatorname{und} \boldsymbol{C}$, and
(ii) a subcategory of $\boldsymbol{C}$, called the **category of weak equivalences** and denoted by $\operatorname{we} \boldsymbol{C}$, of which the maps will be called **weak equivalences**, which are only subject to the requirement that $\operatorname{we} \boldsymbol{C}$ contains all the *objects* of $\boldsymbol{C}$ (and hence also their identity maps).

Similarly a **relative functor** between two relative categories will be a *weak equivalence preserving* functor and a **relative inclusion** $\boldsymbol{A} \to \boldsymbol{B}$ will be a relative functor such that

$$\operatorname{und} \boldsymbol{A} \subset \operatorname{und} \boldsymbol{B} \qquad \text{and} \qquad \operatorname{we} \boldsymbol{A} = \operatorname{we} \boldsymbol{B} \cap \boldsymbol{A}$$

The category of the small relative categories and the relative functors between them will be denoted by **RelCat**. This category comes with an *involution*, i.e., the automorphism

$$Inv : \mathbf{RelCat} \longrightarrow \mathbf{RelCat}$$

which sends each category to its opposite.

Two extreme kinds of relative categories are the

3.2. **Maximal and minimal relative categories.** A relative category will be called

(i) **maximal** if *all* its maps are weak equivalences, and
(ii) **minimal** if the *only* weak equivalences are the *identity maps.*

Given an ordinary category $\boldsymbol{B}$ we will denote by

$$\hat{\boldsymbol{B}} \qquad \text{and} \qquad \check{\boldsymbol{B}}$$

respectively the maximal and the minimal relative categories which have $\boldsymbol{B}$ as their underlying category.

Very useful examples are, for every integer $k \geq 0$, the relative categories

$$\hat{\boldsymbol{k}} \qquad \text{and} \qquad \check{\boldsymbol{k}}$$

where $\boldsymbol{k}$ denotes the k-arrow category

$$0 \longrightarrow \cdots \longrightarrow k$$

For instance we need these to describe the

3.3. **Homotopy relation on RelCat.** Given two objects $\boldsymbol{Y}, \boldsymbol{Z} \in \mathbf{RelCat}$ and two maps $f, g \colon \boldsymbol{Y} \to \boldsymbol{Z} \in \mathbf{RelCat}$, a **strict homotopy** $h \colon f \to g$ will be a natural weak equivalence, i.e., a map

$$h \colon \boldsymbol{Y} \times \hat{\mathbf{1}} \longrightarrow \boldsymbol{Z} \in \mathbf{RelCat}$$

such that (3.2)

$$h(y, 0) = fy \qquad \text{and} \qquad h(y, 1) = gy$$

for every object or map $y \in \boldsymbol{Y}$. More generally, two maps $\boldsymbol{Y} \to \boldsymbol{Z} \in \mathbf{RelCat}$ will be called **homotopic** if they can be connected by a finite zigzag of such strict homotopies.

Similarly a map $f \colon \boldsymbol{Y} \to \boldsymbol{Z} \in \mathbf{RelCat}$ will be called a (strict) **homotopy equivalence** if there exists a map $f' \colon \boldsymbol{Z} \to \boldsymbol{Y} \in \mathbf{RelCat}$ (called a (strict) **homotopy inverse** of f) such that the compositions $f'f$ and ff' are (strictly) homotopic to the identity maps of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ respectively.

A special type of such a strict homotopy equivalence is involved in the definition of

3.4. **Strong deformation retracts.** Given a relative inclusion $\boldsymbol{A} \to \boldsymbol{Z}$ (3.1), $\boldsymbol{A}$ will be called a **strong deformation retract** of $\boldsymbol{Z}$ if there exists a **strong deformation retraction** of $\boldsymbol{Z}$ onto $\boldsymbol{A}$, i.e. a pair (r, s) consisting of

(i) a map $r \colon \boldsymbol{Z} \to \boldsymbol{A} \in \mathbf{RelCat}$, and
(ii) a strict homotopy (3.3) $s \colon r \to 1_{\boldsymbol{Z}}$ such that
(iii) for every object $A \in \boldsymbol{A}$, $rA = A$ and $s \colon rA \to A$ is the identity map of A.

Clearly r *is a strict homotopy equivalence* (3.3) *with the inclusion* $\boldsymbol{A} \to \boldsymbol{Z}$ *as a strict homotopy inverse.*

Using these strong deformation retracts we now define an important class of maps in **RelCat** called

3.5. **Dwyer maps.** In his construction of a model structure on the category of small (ordinary) categories Thomason [T1] introduced *Dwyer maps* which were a kind of neighborhood deformation retracts and recently Cisinski [C] noted the existence of a slightly more general and much more convenient notion which he called *pseudo-Dwyer maps*. Our Dwyer maps will be a relative version of these pseudo-Dwyer maps of Cisinski, i.e.:

A **Dwyer map** will be a map $\boldsymbol{A}' \to \boldsymbol{B} \in \mathbf{RelCat}$ which admits a (unique) factorization

$$\boldsymbol{A}' \approx \boldsymbol{A} \longrightarrow \boldsymbol{B} \in \mathbf{RelCat}$$

in which the first map is an isomorphism and the second is what we will call a **Dwyer inclusion**, i.e. a relative inclusion (3.1) with the following properties:

(i) $\boldsymbol{A}$ is a *sieve* in $\boldsymbol{B}$, i.e. if $f\colon B_1 \to B_2 \in \boldsymbol{B}$ and $B_2 \in \boldsymbol{A}$, then $f \in \boldsymbol{A}$ (or equivalently, if there exists a *characteristic relative functor* $\alpha\colon \boldsymbol{B} \to \hat{\mathbf{1}}$ such that $\alpha^{-1}0 = \boldsymbol{A}$), and

if $Z(\boldsymbol{A}, \boldsymbol{B})$ or just $Z\boldsymbol{A}$ denotes the **cosieve on $\boldsymbol{B}$ generated by $\boldsymbol{A}$**, i.e. the full relative subcategory of $\boldsymbol{B}$ spanned by the objects $B \in \boldsymbol{B}$ for which there exists a map $A \to B \in \boldsymbol{B}$ which $A \in \boldsymbol{A}$ (or equivalently the smallest cosieve in $\boldsymbol{B}$ containing $\boldsymbol{A}$), then

(ii) $\boldsymbol{A}$ is a strong deformation retract of $Z\boldsymbol{A}$ (3.4).

The usefulness of these Dwyer maps is due to the fact that, as we will show in §9, they have such cofibration-like properties as being closed under retracts, pushouts and transfinite compositions.

The definition above of a strong deformation retract, and hence also of a Dwyer map, depends on the choice of the direction of the strict homotopy s in 3.4(iii). The opposite choice yields the notion of a **co-Dwyer map**, i.e., a map obtained from a Dwyer map by replacing the relative categories involved by their opposites.

4. Relative posets and their subdivisions

An important class of relative categories consists of the relative posets and their subdivisions (which are again relative posets).

With each relative poset $\boldsymbol{P}$ one can associate two subdivisions, a "terminal" subdivision $\xi_t\boldsymbol{P}$ and an "initial" subdivision $\xi_i\boldsymbol{P}$. Unlike the corresponding subdivisions of ordinary posets, these subdivisions care in general not each others opposites, but merely each others "conjugates" in the sense that there are canonical isomorphisms

$$(\xi_i\boldsymbol{P})^{\mathrm{op}} \approx \xi_t(\boldsymbol{P}^{\mathrm{op}}) \qquad \text{or equivalently} \qquad (\xi_t\boldsymbol{P})^{\mathrm{op}} \approx \xi_i(\boldsymbol{P}^{\mathrm{op}})$$

For instance, if $\boldsymbol{P} = \check{\mathbf{2}}$ (3.2) and $\xrightarrow{\sim}$ indicates a weak equivalence, then

$$\xi_t\check{\mathbf{2}} \qquad = \qquad \begin{tikzcd}
& & 1 \arrow[dl, "\sim"'] \arrow[dr] \arrow[dd] & & \\
& 01 \arrow[dr] & & 12 \arrow[dl, "\sim"] & \\
& & 012 & & \\
0 \arrow[uur] \arrow[urr] \arrow[rr] & & 02 \arrow[u, "\sim"'] & & 2 \arrow[uul, "\sim"'] \arrow[ull, "\sim"] \arrow[ll, "\sim"]
\end{tikzcd}$$

while

$$\xi_i\check{\mathbf{2}} \qquad = \qquad \begin{tikzcd}
& & 1 & & \\
& 01 \arrow[ur] \arrow[dl, "\sim"'] & & 12 \arrow[ul, "\sim"'] \arrow[dr] & \\
& & 012 \arrow[uu] \arrow[ul, "\sim"] \arrow[ur] \arrow[dll, "\sim"'] \arrow[d, "\sim"'] \arrow[drr] & & \\
0 & & 02 \arrow[ll, "\sim"] \arrow[rr] & & 2
\end{tikzcd}$$

In more detail:

4.1. **Relative posets.** A **relative poset** is a relative category $\boldsymbol{P}$ such that und $\boldsymbol{P}$ (3.1) (and hence we $\boldsymbol{P}$) is a poset. The resulting full subcategory of **RelCat** spanned by these relative posets will be denoted by **RelPos**.

4.2. **Terminal and initial subdivisions.** The **terminal** (resp. **initial**) **subdivision** of a relative poset $\boldsymbol{P}$ will be the relative poset $\xi_t\boldsymbol{P}$ (resp. $\xi_i\boldsymbol{P}$) which has

(i) as objects the *monomorphisms* (3.2)

$$\check{\boldsymbol{n}} \longrightarrow \boldsymbol{P} \in \mathbf{RelPos} \qquad (n \geq 0)$$

(ii) as maps

$$\begin{aligned} &(x_1 \colon \check{\boldsymbol{n}}_1 \to \boldsymbol{P}) \longrightarrow (x_2 \colon \check{\boldsymbol{n}}_2 \to \boldsymbol{P}) \\ (\text{resp.} \qquad &(x_2 \colon \check{\boldsymbol{n}}_2 \to \boldsymbol{P}) \longrightarrow (x_1 \colon \check{\boldsymbol{n}}_1 \to \boldsymbol{P})) \end{aligned}$$

the commutative diagrams of the form

$$\begin{tikzcd} \check{\boldsymbol{n}}_1 \arrow[rr] \arrow[dr, "x_1"'] & & \check{\boldsymbol{n}}_2 \arrow[dl, "x_2"] \\ & \boldsymbol{P} & \end{tikzcd}$$

and

(iii) as weak equivalences those of the above (ii) diagrams for which the induced map $x_1n_1 \to x_2n_2$ (resp. $x_20 \to x_10$) is a weak equivalence in $\boldsymbol{P}$.

This terminal (resp. initial) subdivision comes with a **terminal** (resp. **initial**) **projection functor**

$$\pi_t \colon \xi_t\boldsymbol{P} \longrightarrow \boldsymbol{P} \qquad (\text{resp.} \pi_i \colon \xi_i\boldsymbol{P} \longrightarrow \boldsymbol{P})$$

which sends an object $x\colon \check{\boldsymbol{n}} \to \boldsymbol{P} \in \xi_t\boldsymbol{P}$ (resp. $\xi_i\boldsymbol{P}$) to the object $xn \in \boldsymbol{P}$ (resp. $x0 \in \boldsymbol{P}$) and a commutative triangle as above to the map $x_1n_1 \to x_2n_2 \in \boldsymbol{P}$ (resp. $x_20 \to x_10 \in \boldsymbol{P}$), which clearly implies that

(iv) *a map in $\xi_t\boldsymbol{P}$ (resp. $\xi_i\boldsymbol{P}$) is a weak equivalence iff its image under π_t (resp. π_i) is so in $\boldsymbol{P}$.*

We also note the

4.3. **Naturality of the subdivisions.** One readily verifies that the above functions ξ_t and ξ_i on the objects of **RelPos** can be extended to functors

$$\xi_t, \xi_i \colon \mathbf{RelPos} \longrightarrow \mathbf{RelPos}$$

by sending, for a map $f\colon \boldsymbol{P} \to \boldsymbol{P}' \in \mathbf{RelPos}$ every monomorphism $\check{\boldsymbol{n}} \to \boldsymbol{P}$ to the unique monomorphism $\check{\boldsymbol{n}}' \to \boldsymbol{P}'$ for which there exists a commutative diagram of the form

$$\begin{tikzcd} \check{\boldsymbol{n}} \arrow[r] \arrow[d] & \check{\boldsymbol{n}}' \arrow[d] \\ \boldsymbol{P} \arrow[r, "f"'] & \boldsymbol{P}' \end{tikzcd}$$

in which the top map is an epimorphism.

Next we verify

4.4. **The conjugation.** To verify the conjugation mentioned at the beginning of this section we note that, using the unique isomorphisms

$$\boldsymbol{n} \approx \boldsymbol{n}^{\mathrm{op}} \qquad (n \geq 0),$$

one can construct an isomorphism $\operatorname{und}(\xi_i \boldsymbol{P})^{\mathrm{op}} \xrightarrow{\approx} \operatorname{und} \xi_t(\boldsymbol{P}^{\mathrm{op}})$ by associating with each map

$$\begin{tikzcd} \check{\boldsymbol{n}}_1 \arrow[rr] \arrow[dr, "y_1"'] & & \check{\boldsymbol{n}}_2 \arrow[dl, "y_2"] \\ & \boldsymbol{P}^{\mathrm{op}} & \end{tikzcd} \qquad \text{in } \xi_t(\boldsymbol{P}^{\mathrm{op}})$$

the map

$$\begin{tikzcd} \check{\boldsymbol{n}}_1 \arrow[rr] \arrow[d, "\approx"'] & & \check{\boldsymbol{n}}_2 \arrow[d, "\approx"] \\ \check{\boldsymbol{n}}_1^{\mathrm{op}} \arrow[dr, "y_1^{\mathrm{op}}"'] & & \check{\boldsymbol{n}}_2^{\mathrm{op}} \arrow[dl, "y_2^{\mathrm{op}}"] \\ & \boldsymbol{P} & \end{tikzcd} \qquad \text{in } (\xi_i \boldsymbol{P})^{\mathrm{op}}$$

A rather straightforward calculation yields that this isomorphism is actually an isomorphism of relative posets.

We end with some

4.5. **Final comments.**

(i) The reason that, given a relative poset $\boldsymbol{P}$, we considered in this section both its terminal and its initial subdivision is that, as will be shown in 9.4–9.6 below, in order to obtain the needed Dwyer maps we need the **two-fold subdivision** $\xi_t \xi_i \boldsymbol{P}$ and not, as one might have expected from Thomason's original result the iterated subdivisions $\xi_t^2 \boldsymbol{P}$ or $\xi_i^2 \boldsymbol{P}$. It will therefore be convenient to denote the two-fold subdivision

$$\xi_t \xi_i \boldsymbol{P} \qquad \text{by } \xi \boldsymbol{P}$$

and the associated composition

$$\xi_t \xi_i \boldsymbol{P} \xrightarrow{\pi_t} \xi_i \boldsymbol{P} \xrightarrow{\pi_i} \boldsymbol{P} \qquad \text{by} \qquad \xi \boldsymbol{P} \xrightarrow{\pi} \boldsymbol{P}.$$

That Thomason did not have to do this is due to the fact that *if* $\boldsymbol{P}$ *is maximal* (3.2)*, then there are obvious isomorphisms*

$$\xi_t^2 \boldsymbol{P} \approx \xi \boldsymbol{P} \qquad \text{and} \qquad \xi_i^2 \boldsymbol{P} \approx \xi \boldsymbol{P}.$$

(ii) Dually, there is a **conjugate two-fold subdivision** $\xi_i \xi_t$, which we denote by $\overline{\xi} \boldsymbol{P}$, and for which we denote the associated composition

$$\xi_i \xi_t \boldsymbol{P} \xrightarrow{\pi_i} \xi_t \boldsymbol{P} \xrightarrow{\pi_t} \boldsymbol{P} \qquad \text{by} \qquad \overline{\xi} \boldsymbol{P} \xrightarrow{\overline{\pi}} \boldsymbol{P}.$$

(iii) Given a relative poset $\boldsymbol{P}$ it is sometimes convenient to denote an object

$$x \colon \check{\boldsymbol{n}} \longrightarrow \boldsymbol{P} \in \xi_t \boldsymbol{P} \quad \text{or} \quad \xi_i \boldsymbol{P}$$

by the sequence

$$(x0, \ldots, xn)$$

of objects of $\boldsymbol{P}$.

5. Some more preliminaries

To formulate our main result (in 6.1 below) we need

(i) a description of what is meant by *lifting* a cofibrantly generated model structure,
(ii) the *Reedy model structure* on the category s$\boldsymbol{S}$ of bisimplicial sets as well as its *left Bousfield localizations*,
(iii) two adjunctions s$\boldsymbol{S} \leftrightarrow$ **RelCat**, and
(iv) a *key lemma*.

We thus start with

5.1. **Lifting model structures.** ([H, sec. 11.3]) Given a cofibrantly generated model category $\boldsymbol{F}$ and an adjunction

$$f \colon \boldsymbol{F} \longleftrightarrow \boldsymbol{G} : g$$

one says that the model structure on $\boldsymbol{F}$ **lifts** to a cofibrantly generated model structure on $\boldsymbol{G}$ if

(i) the sets of the images in $\boldsymbol{G}$ under the left adjoint f of some choice of generating cofibrations and generating trivial cofibrations of the model structure on $\boldsymbol{F}$ *admit the small object argument*, and
(ii) the right adjoint g takes all (possibly transfinite) compositions of pushouts of the images in $\boldsymbol{G}$ under f of the generating trivial cofibrations of $\boldsymbol{F}$ to weak equivalences in $\boldsymbol{F}$,

in which case

(iii) the *generating cofibrations* and *generating trivial cofibrations* of the model structure on $\boldsymbol{G}$ are the images under f of the generating cofibrations and generating trivial cofibrations of the model structure on $\boldsymbol{F}$, and
(iv) a map in $\boldsymbol{G}$ is a *weak equivalence* or a *fibration* iff its image under g is so in $\boldsymbol{F}$.

Next we recall

5.2. **The Reedy model structure on s$\boldsymbol{S}$ and its left Bousfield localizations.** As usual let $\boldsymbol{\Delta} \subset$ **Cat** (the category of small categories) be the full subcategory spanned by the posets $\boldsymbol{n}$ ($n \geq 0$) (3.2) and let $\boldsymbol{S}$ and s$\boldsymbol{S}$ denote the resulting categories

$$\boldsymbol{S} = \mathbf{Set}^{\boldsymbol{\Delta}^{\mathrm{op}}} \qquad \text{and} \qquad \mathrm{s}\boldsymbol{S} = \mathbf{Set}^{\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}}$$

of *simplicial* and *bisimplicial* sets.

(i) The *standard model structure* on $\boldsymbol{S}$ is the cofibrantly generated proper model structure ([H, pp. 210 and 239]) in which
   (a) the cofibrations are the monomorphisms, and
   (b) the weak equivalences are the maps which induce homotopy equivalences between the geometric realizations.

(ii) The resulting *Reedy model structure* on $\mathrm{s}\boldsymbol{S}$ is the cofibrantly generated proper model structure in which
  (a) the cofibrations are the monomorphisms, and
  (b) the weak equivalences are the *Reedy (weak) equivalences*, i.e. the maps $L \to M \in \mathrm{s}\boldsymbol{S}$ for which the restrictions

$$L(\boldsymbol{p}, -) \longrightarrow M(\boldsymbol{p}, -) \in \boldsymbol{S} \qquad (p \geq 0)$$

  are weak equivalences (i).

(iii) A *left Bousfield localization* ([H, p. 57]) of this Reedy structure is any cofibrantly generated left proper model structure in which
  (a) the cofibrations are the monomorphisms, and
  (b) the weak equivalences *include* the Reedy equivalences.

(v) We note that the category $\mathrm{s}\boldsymbol{S}$ admits an *involution*

$$Inv : \mathrm{s}\boldsymbol{S} \longrightarrow \mathrm{s}\boldsymbol{S},$$

which is the automorphism that sends an object $L \in \mathrm{s}\boldsymbol{S}$ — i.e., a functor $\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ — to the composition

$$\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{\sigma^{\mathrm{op}} \times \sigma^{\mathrm{op}}} \boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{L} \mathbf{Set},$$

wherein $\sigma : \boldsymbol{\Delta} \longrightarrow \boldsymbol{\Delta}$ denotes the unique nontrivial automorphism of Δ.

We also need

5.3. **Two adjunctions.** Let $\Delta[m,n] \in \mathrm{s}\boldsymbol{S}$ $(m,n \geq 0)$ denote the *standard* (m,n)-*bisimplex* which has as its (i,j)-bisimplices $(i,j \geq 0)$ the maps $(\boldsymbol{i},\boldsymbol{j}) \to (\boldsymbol{m},\boldsymbol{n}) \in \boldsymbol{\Delta} \times \boldsymbol{\Delta}$ (5.2). Our main result then involves the adjunctions

$$K \colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N \qquad \text{and} \qquad K_\xi \colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N_\xi,$$

in which K and K_ξ are the colimit preserving functors which send $\Delta[p,q]$ $(p,q \geq 0)$ to the relative categories (3.2 and 4.5(i))

$$\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \qquad \text{and} \qquad \xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}),$$

respectively, and N and N_ξ send an object $\boldsymbol{X} \in \mathbf{RelCat}$ to the bisimplicial sets which have as their (p,q)-bisimplices $(p,q \geq 0)$ the maps

$$\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \longrightarrow \boldsymbol{X} \qquad \text{and} \qquad \xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \longrightarrow \boldsymbol{X} \in \mathbf{RelCat},$$

respectively.

The most important of these functors is the functor N which Charles Rezk called the **classifying diagram**, but which is now often referred to as the **(simplicial) nerve functor**. It is connected to the functor N_ξ by a natural transformation

$$\pi^* \colon N \longrightarrow N_\xi$$

induced by the natural transformation $\pi \colon \xi \to \mathrm{id}$ (4.5(i)). This natural transformation π^* is of particular importance as, in view of the following key lemma 5.4, it enables us, in the proof of theorem 6.1 below, to use the functor N instead of the much more cumbersome functor N_ξ.

5.4. **A key lemma. The natural transformation $\pi^* \colon N \to N_\xi$ is a natural Reedy equivalence** (5.2). A proof will be given in §§7-8.

6. A statement of the main results

Our main result is

6.1. **Theorem. Lifting model structures on s$\boldsymbol{S}$ to Quillen equivalent ones on RelCat.** *The adjunction* (5.3)

$$K_\xi \colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N_\xi$$

lifts (5.1) *every left Bousfield localization of the Reedy model structure on* s$\boldsymbol{S}$ *(and in particular Rezk's complete Segal structure) to a Quillen equivalent cofibrantly generated left proper model structure on* **RelCat** *in which*

- (i) *a map is a weak equivalence iff its image under N_ξ (or equivalently* (5.4) *iff its image under N) is so in* s$\boldsymbol{S}$,
- (ii) *a map is a fibration iff its image under N_ξ is so in* s$\boldsymbol{S}$,
- (iii) *every cofibration is a Dwyer map* (3.5),
- (iv) *every cofibrant object is a relative poset* (4.1).

Moreover, the model structure lifted from the Reedy structure itself is also right proper.

A proof will be given in §10.

Dualizing the proof of both 5.4 and 6.1, one obtains the following

6.2. **Theorem. The conjugate model structures on RelCat.** *The key lemma* 5.4 *and the theorem* 6.1 *remain valid if one replaces*

- (i) ξ *with* $\overline{\xi}$ (4.5(ii)),
- (ii) π *with* $\overline{\pi}$ (4.5(ii)), *and*
- (iii) *the phrase* Dwyer map *with the phrase* co-Dwyer map (3.5).

6.3. **Corollary.** *The two model structures on* **RelCat** *lifted, as in* 6.1 *and* 6.2, *from the Reedy model structure on* s$\boldsymbol{S}$ *or any left Bousfield localization thereof*

- (i) *are Quillen equivalent,*
- (ii) *have the same weak equivalences, and hence*
- (iii) *have identical underlying relative categories.*

6.4. **Theorem. The involution of RelCat.**

- (i) *The involution* (3.1)

$$Inv : \mathbf{RelCat} \longrightarrow \mathbf{RelCat}$$

  is an isomorphism between the two model structures of (6.3).
- (ii) *Equivalently, a map $f \in$* **RelCat** *is a cofibration, fibration, or weak equivalence in one of those model structures iff $Inv(f) \in$* **RelCat** *is so in the other.*

Proof. 6.4(ii) follows readily from the existence, for every pair of integers $p, q \geq 0$, of an isomorphism

$$\overline{\xi}(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \approx (\xi(\check{\boldsymbol{p}}^{\mathrm{op}} \times \hat{\boldsymbol{q}}^{\mathrm{op}}))^{\mathrm{op}} \approx (\xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}))^{\mathrm{op}},$$

in which the first isomorphism is as in 4.5(i), and the second is induced by the *unique* isomorphism

$$\check{\boldsymbol{p}}^{\mathrm{op}} \times \hat{\boldsymbol{q}}^{\mathrm{op}} \approx \check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}.$$

6.5. **Some $(\infty,1)$-categorical comments on the Rezk case.** For the purposes of this section, let **RelCat** and s$\boldsymbol{S}$ denote the relative categories in which the weak equivalences are the Rezk ones, and denote by **RELCAT** the similarly defined large relative category. Then clearly

(i) *as* s$\boldsymbol{S}$ *is a model for the theory of* $(\infty,1)$*-categories, so is* **RelCat**.

To make a similar statement for the involution *Inv* : **RelCat** $\longrightarrow$ **RelCat** 3.1, let L^H denote the hammock localization of [DK]. Then one can, for every relative category $\boldsymbol{X}$, define the **space** haut $\boldsymbol{X}$ of auto-equivalences of $\boldsymbol{X}$ as the space which consists of the invertible components of the function space

$$L^H\mathbf{RELCAT}(\boldsymbol{X},\boldsymbol{X}).$$

It then follows from a result of Toën [T2, 6.3], that the space haut **RelCat** *has two components, which are both contractible.* One of these contains the identity map of **RelCat**, and thus the vertices of the other are the *nontrivial auto-equivalences* of **RelCat**.

Now we can state that

(ii) *the involution Inv* : **RelCat** $\longrightarrow$ **RelCat** (3.1) *is a nontrivial auto-equivalence of* **RelCat**, *and hence it models the contractible space of the nontrivial auto-equivalences of theories of* $(\infty,1)$*-categories.*

Proof. This follows readily from

(i) the observation of Toën [T2, 6.3] that the involution *Inv* : s$\boldsymbol{S}$ $\longrightarrow$ s$\boldsymbol{S}$ (5.2(v)) is an automorphism of relative categories and is a nontrivial auto-equivalence of s$\boldsymbol{S}$, and
(ii) the commutativity of the diagram

$$\begin{tikzcd} \mathbf{RelCat} \arrow[r, "Inv"] \arrow[d, "N"'] & \mathbf{RelCat} \arrow[d, "N"] \\ \mathrm{s}\boldsymbol{S} \arrow[r, "Inv"'] & \mathrm{s}\boldsymbol{S}. \end{tikzcd}$$

To next deal with Thomason's result [T1] in our language we need

6.6. **Two more adjunctions.** Let $\widehat{\mathbf{Cat}} \subset$ **RelCat** denote the full subcategory spanned by the *maximal* (3.2) relative categories. Then one has, corresponding to the adjunctions of 5.3, adjunctions

$$k\colon \boldsymbol{S} \longleftrightarrow \widehat{\mathbf{Cat}} : n \qquad \text{and} \qquad k_\xi\colon \boldsymbol{S} \longleftrightarrow \widehat{\mathbf{Cat}} : n_\xi$$

in which respectively k and k_ξ are the colimit preserving functors which send the standard simplex $\Delta[q]$ $(q \ge 0)$ to the maximal relative categories

$$\hat{\boldsymbol{q}} \qquad \text{and} \qquad \xi\hat{\boldsymbol{q}}$$

and n and n_ξ send an object $Y \in \widehat{\mathbf{Cat}}$ to the simplicial sets which have as its q-simplices $(q \ge 0)$ the maps

$$\hat{\boldsymbol{q}} \longrightarrow \boldsymbol{Y} \qquad \text{and} \qquad \xi\hat{\boldsymbol{q}} \longrightarrow \boldsymbol{Y} \in \widehat{\mathbf{Cat}} \subset \mathbf{RelCat}$$

The functor $n\colon \widehat{\mathbf{Cat}} \to \boldsymbol{S}$ is the (classical) **nerve** functor and is connected to the functor $n_\xi\colon \widehat{\mathbf{Cat}} \to \boldsymbol{S}$ by a natural transformation

$$\pi^*\colon n \longrightarrow n_\xi$$

induced by the natural transformation $\pi\colon \xi \to \mathrm{id}$ (4.5(i)).

In our language Thomason's result then becomes

6.7. **Thomason's theorem** [T1]. *The adjunction*

$$k_\xi\colon \boldsymbol{S} \longleftrightarrow \widehat{\mathbf{Cat}} : n_\xi$$

lifts (5.1) *the standard model structure on* $\boldsymbol{S}$ (5.2) *to a Quillen equivalent cofibrantly generated proper model structure on* $\widehat{\mathbf{Cat}}$ *in which*

(i) *a map is a weak equivalence or a fibration iff its image under* n_ξ *is so in* $\boldsymbol{S}$,
(ii) *every cofibration is a Dwyer map* (3.5)*, and*
(iii) *every cofibrant object is a relative poset* (4.1).

Moreover

(iv) *the natural transformation* $\pi^*\colon n \to n_\xi$ *is a natural weak equivalence*

and hence

(v) *a map is also a weak equivalence iff its image under the nerve functor* n *is so in* $\boldsymbol{S}$.

We end this section pointing out a tight connection between our result and Thomason's original one [T1].

6.8. **A tight connection between theorems 6.1 and 6.7.** If one considers the category $\boldsymbol{S}$ as the subcategory of the category $\mathrm{s}\boldsymbol{S}$ consisting of the bisimplicial sets L for which

$$L(\boldsymbol{p},\boldsymbol{q}) = L(\boldsymbol{0},\boldsymbol{q}) \qquad \text{for all } p, q \geq 0$$

then the inclusions

$$\boldsymbol{S} \subset \mathrm{s}\boldsymbol{S} \qquad \text{and} \qquad \widehat{\mathbf{Cat}} \subset \mathbf{RelCat}$$

are the left adjoints in adjunctions

$$i\colon \boldsymbol{S} \longleftrightarrow \mathrm{s}\boldsymbol{S} : r \qquad \text{and} \qquad i\colon \widehat{\mathbf{Cat}} \longleftrightarrow \mathbf{RelCat} : r$$

for which the units $1 \to ri$ are both the identity natural transformations. Then one readily verifies that

6.9. **Proposition.** *The diagram*

$$\begin{tikzcd}
\boldsymbol{S} \arrow[r, shift left, "k_\xi"] \arrow[d, shift right, "i"'] & \widehat{\mathbf{Cat}} \arrow[l, shift left, "n_\xi"] \arrow[d, shift left, "i"] \\
\mathrm{s}\boldsymbol{S} \arrow[u, shift right, "r"'] \arrow[r, shift right, "K_\xi"'] & \mathbf{RelCat} \arrow[l, shift right, "N_\xi"'] \arrow[u, shift left, "r"]
\end{tikzcd}$$

in which the outside arrows are the left adjoints and the inside ones the right adjoints has the following properties:

(i) *The horizontal adjunctions are both Quillen equivalences (*6.1 *and* 6.7*) and the vertical adjunctions are both Quillen pairs.*
(ii) *The diagram commutes as a square of adjunctions and as a square of Quillen pairs.*

Moreover

(iii) $k_\xi = rK_\xi i$ *and* $n_\xi = rN_\xi i$.

7. Some homotopy preserving functors

In preparation for the proof (in §8 below) of lemma 5.4 we here

(i) note that the category **RelCat** is *cartesian closed* and that the homotopy relation on **RelCat** is compatible with this cartesian closure,
(ii) prove that the subdivision functors (§4) preserve homotopies between *finite* relative posets and
(iii) describe a sufficient condition on a functor $\mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$ (5.2) in order that it sends homotopic maps in **RelCat** to homotopic maps in $\mathrm{s}\boldsymbol{S}$.

We thus start with

7.1. **Cartesian closure of RelCat.** *The category* **RelCat** *is cartesian closed.* That is [M, Ch. IV, sec. 6], we have the following.

(i) For every object $\boldsymbol{Y} \in \mathbf{RelCat}$, the functor

$$- \times \boldsymbol{Y} \colon \mathbf{RelCat} \longrightarrow \mathbf{RelCat}$$

has a right adjoint $(-)^{\boldsymbol{Y}}$, which associates with an object $\boldsymbol{Z} \in \mathbf{RelCat}$ the **relative category of relative functors** $\boldsymbol{Z}^{\boldsymbol{Y}}$, which has
  (a) as *objects* the maps $\boldsymbol{Y} \to \boldsymbol{Z} \in \mathbf{RelCat}$, and
  (b) as *maps* and *weak equivalences* respectively the maps (3.2)

$$\boldsymbol{Y} \times \check{\mathbf{1}} \longrightarrow \boldsymbol{Z} \qquad \text{and} \qquad \boldsymbol{Y} \times \hat{\mathbf{1}} \longrightarrow \boldsymbol{Z} \qquad \in \mathbf{RelCat}.$$

(ii) For every three objects $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z} \in \mathbf{RelCat}$, there is [M, Ch. IV, sec. 6, Ex. 3] a natural *enriched adjunction isomorphism*

$$\boldsymbol{Z}^{\boldsymbol{X} \times \boldsymbol{Y}} \approx (\boldsymbol{Z}^{\boldsymbol{Y}})^{\boldsymbol{X}} \in \mathbf{RelCat},$$

which sends
  (a) a map $f \colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Z}$ to the map $g \colon \boldsymbol{X} \to \boldsymbol{Z}^{\boldsymbol{Y}}$, which sends an object $x \in \boldsymbol{X}$ to the map $gx \colon \boldsymbol{Y} \to \boldsymbol{Z}$ given by $(gx)y = f(x,y)$ for every object $y \in \boldsymbol{Y}$, and
  (b) a map

$$\boldsymbol{X} \times \boldsymbol{Y} \times \check{\mathbf{1}} \longrightarrow \boldsymbol{Z} \qquad (\text{resp. } \boldsymbol{X} \times \boldsymbol{Y} \times \hat{\mathbf{1}} \longrightarrow \boldsymbol{Z})$$

  to the map

$$\boldsymbol{X} \times \check{\mathbf{1}} \longrightarrow \boldsymbol{Z}^{\boldsymbol{Y}} \qquad (\text{resp. } \boldsymbol{X} \times \hat{\mathbf{1}} \longrightarrow \boldsymbol{Z}^{\boldsymbol{Y}})$$

  obtained from the obvious composition

$$\boldsymbol{X} \times \check{\mathbf{1}} \times \boldsymbol{Y} \approx \boldsymbol{X} \times \boldsymbol{Y} \times \check{\mathbf{1}} \longrightarrow \boldsymbol{Z} \qquad (\text{resp. } \boldsymbol{X} \times \hat{\mathbf{1}} \times \boldsymbol{Y} \approx \boldsymbol{X} \times \boldsymbol{Y} \times \hat{\mathbf{1}} \longrightarrow \boldsymbol{Z}).$$

7.2. **Proposition.** *If two maps* $f, g \colon \boldsymbol{X} \to \boldsymbol{Y} \in \mathbf{RelCat}$ *are strictly homotopic* (3.3)*, then so are, for every object* $\boldsymbol{Z} \in \mathbf{RelCat}$ *the induced maps* (7.1)

$$f^*, g^* \colon \boldsymbol{Z}^{\boldsymbol{Y}} \longrightarrow \boldsymbol{Z}^{\boldsymbol{X}},$$

and hence, if $e \colon \boldsymbol{X} \to \boldsymbol{Y} \in \mathbf{RelCat}$ *is a (strict) homotopy equivalence* (3.3)*, then so is, for every object* $\boldsymbol{Z} \in \mathbf{RelCat}$*, the induced maps*

$$e^* \colon \boldsymbol{Z}^{\boldsymbol{Y}} \longrightarrow \boldsymbol{Z}^{\boldsymbol{X}}.$$

Proof. Given a strict homotopy $h\colon \boldsymbol{X} \times \hat{\boldsymbol{1}} \to \boldsymbol{Y}$, the desired strict homotopy is the map $\boldsymbol{Z}^{\boldsymbol{Y}} \times \hat{\boldsymbol{1}} \to \boldsymbol{Z}^{\boldsymbol{X}}$ which is adjoint (7.1(ii)a) to the composition

$$\boldsymbol{Z}^{\boldsymbol{Y}} \xrightarrow{z^h} \boldsymbol{Z}^{(\boldsymbol{X}\times\hat{\boldsymbol{1}})} \approx (\boldsymbol{Z}^{\boldsymbol{X}})^{\hat{\boldsymbol{1}}}$$

in which the isomorphism is as in 7.1(ii)a.

7.3. **Proposition.** *The subdivision functors ξ_t, ξ_i and $\xi = \xi_t\xi_i$ (§4)*

(i) *preserve homotopies between maps from finite relative posets*

and hence also

(ii) *preserve homotopy equivalences between finite relative posets.*

In particular,

(iii) *for every pair of integers $p, q \geq 0$ all maps in the commutative diagram*

$$\begin{tikzcd}
\xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) = \xi_t\xi_i(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \arrow[r, "\pi_t\xi_i"] \arrow[d] & \xi_i(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \arrow[r, "\pi_i"] \arrow[d] & \check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \arrow[d] \\
\xi\check{\boldsymbol{p}} = \xi_t\xi_i\check{\boldsymbol{p}} \arrow[r, "\pi_t\xi_i"] & \xi_i\check{\boldsymbol{p}} \arrow[r, "\pi_i"] & \check{\boldsymbol{p}},
\end{tikzcd}$$

in which the vertical maps are induced by the projection $\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \to \check{\boldsymbol{p}}$, are homotopy equivalences.

Proof. We first deduce (iii) from (ii).

To do this we note that the map $\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \to \check{\boldsymbol{p}}$ is obviously a homotopy equivalence; hence, in view of (ii), so are the other two vertical maps.

Next we consider the commutative diagram

$$\begin{tikzcd}
\xi_t\xi_i\check{\boldsymbol{p}} \arrow[r, "\pi_t\xi_i"] \arrow[d, "\xi_t\pi_i"'] & \xi_i\check{\boldsymbol{p}} \arrow[d, "\pi_i"] \\
\xi_t\check{\boldsymbol{p}} \arrow[r, "\pi_t"] & \check{\boldsymbol{p}},
\end{tikzcd}$$

for which one readily verifies that the maps going to $\check{\boldsymbol{p}}$ are homotopy equivalences with as homotopy inverses the maps which send an object $i \in \check{\boldsymbol{p}}$ to the objects (4.5(iii))

$$(0, \ldots, i) \in \xi_t\check{\boldsymbol{p}} \qquad \text{and} \qquad (p-i, \ldots, p) \in \xi_i\check{\boldsymbol{p}}$$

respectively and the desired result now follows from the observation that, in view of (ii), the map $\xi_t\pi_i$ is a weak equivalence and thus so is the map $\pi_t\xi_i$.

Next we note that (ii) follows from (i). It thus remains to prove (i).

To do this, it suffices to observe that, for every finite relative poset $\boldsymbol{P}$, if

(i) n is the number of objects of $\boldsymbol{P}$ and one denotes the objects of $\boldsymbol{P}$ by the integers $1, \ldots, n$ in such a manner that, for every two such integers a and b one has $a \leq b$, whenever there exists a map $a \to b \in \boldsymbol{P}$, and

(ii) $\boldsymbol{J}$ denotes the *maximal* relative poset (3.2) which has $2n + 1$ objects $j_0, \ldots, j_{2n}$ and, for every integer i with $0 \leq i \leq n-1$, maps

$$j_{2i} \longrightarrow j_{2i+1} \longleftarrow j_{2i+2},$$

then we have the following.

(i) *There exists a map*

$$k\colon \xi_t\boldsymbol{P}\times\boldsymbol{J}\longrightarrow \xi_t(\boldsymbol{P}\times\hat{\mathbf{1}})\in\mathbf{RelPos}$$

such that, in the notation of 4.5(iii), *for every object* $(r_1,\dots,r_u)\in\xi_t\boldsymbol{P}$

$$k\big((r_1,\dots,r_u),j_{2n}\big)=\big((r_1,0),\dots,(r_u,0)\big)$$

and

$$k\big((r_1,\dots,r_u),j_0\big)=\big((r_1,1),\dots,(r_u,1)\big).$$

For in that case,

(ii) *for any two maps* $f,g\colon\boldsymbol{P}\to\boldsymbol{X}\in\mathbf{RelCat}$ *and strict homotopy* (3.3)

$$h\colon\boldsymbol{P}\times\hat{\mathbf{1}}\longrightarrow\boldsymbol{X}\in\mathbf{RelCat}$$

between them, the composition

$$\xi_t\boldsymbol{P}\times\boldsymbol{J}\xrightarrow{k}\xi_t(\boldsymbol{P}\times\hat{\mathbf{1}})\xrightarrow{\xi_t h}\xi_t\boldsymbol{X}$$

is a homotopy between $\xi_t f$ *and* $\xi_t q$.

A lengthy but essentially straightforward calculation (which we will leave to the reader) then yields that

(iii) *such a map k can be obtained by defining, for every integer i with $0\le i\le n$ and every object* $(p_1,\dots,p_s,q_1,\dots,q_t)\in\xi_t\boldsymbol{P}$ *with* $p_s<i\le q_1$,

$$k\big((p_1,\dots,p_s,q_1,\dots,q_t),j_{21}\big)=\big((p_1,0),\dots,(p_s,0),(q_1,1),\dots,(q_t,1)\big),$$

and, for every integer i with $0\le i\le n-p$ and object $(p_1,\dots,p_s,q_1,\dots,q_t)\in\xi_t\boldsymbol{P}$ *with* $p_s<i<q_t$,

$$k\big((p_1,\dots,p_s,q_1,\dots,q_t),j_{2i+1}\big)=\big((p_1,0),\dots,(p_s,0),(q_1,1),\dots,(q_t,1)\big)$$

and

$$k\big((p_1,\dots,p_s,i,q_1,\dots,q_s),j_{2i+1}\big)=\big((p_1,0),\dots,(p_s,0),(i,0),(i,1),(q_1,1),\dots,(q_t,1)\big).$$

It thus remains to describe the needed sufficient condition on a functor $\mathbf{RelCat}\to \mathrm{s}\boldsymbol{S}$ (5.2) in order that it preserve homotopies, and for this we better first make clear what exactly we will mean by

7.4. Homotopic maps and homotopy equivalences in s$\boldsymbol{S}$. We will call

(i) two maps $A\to B\in\mathrm{s}\boldsymbol{S}$ **homotopic** if they can be connected by a finite zigzag of maps of the form $A\times\Delta[0,1]\to B\in\mathrm{s}\boldsymbol{S}$, and

(ii) a map $f\colon A\to B\in\mathrm{s}\boldsymbol{S}$ a **homotopy equivalence** if there exists a map $g\colon B\to A\in\mathrm{s}\boldsymbol{S}$ (called a **homotopy inverse** of f) such that the compositions gf and fg are homotopic to the identity maps of A and B respectively.

These definitions clearly imply that

(iii) *every homotopy equivalence in* s$\boldsymbol{S}$ *is a Reedy equivalence* (5.2).

Next, for every functor $\alpha\colon \mathbf{\Delta}\times\mathbf{\Delta}\to\mathbf{RelCat}$, let $N_\alpha\colon \mathbf{RelCat}\to \mathrm{s}\boldsymbol{S}$ denote the functor that to every object $\boldsymbol{X}\in\mathbf{RelCat}$ and to every pair of integers $p,q\ge 0$ assigns the set of maps $\alpha(\boldsymbol{p},\boldsymbol{q})\to\boldsymbol{X}\in\mathbf{RelCat}$. Then one has:

7.5. **Proposition.** *If* $\iota\colon \mathbf{\Delta}\times\mathbf{\Delta}\to\mathbf{RelCat}$ (5.2) *is the functor that sends* $(\boldsymbol{p},\boldsymbol{q})$ *to* $\hat{\boldsymbol{q}}$ *(*$p,q\ge 0$*), and* $\alpha\colon \mathbf{\Delta}\times\mathbf{\Delta}\to\mathbf{RelCat}$ *is a functor for which there exists a natural transformation* $\varepsilon\colon \alpha\to\iota$*, then the functor* $N_\alpha\colon\mathbf{RelCat}\to\mathrm{s}\boldsymbol{S}$ *sends homotopic maps in* $\mathbf{RelCat}$ *to homotopic maps in* $\mathrm{s}\boldsymbol{S}$ (7.4)*, and hence homotopy equivalences in* $\mathbf{RelCat}$ *to homotopy equivalences in* $\mathrm{s}\boldsymbol{S}$*.*

This is in particular the case if

(i) $\alpha=\iota$ *and* $\varepsilon=\mathrm{id}$

and, for every pair of integers $p,q\ge 0$*, if*

(ii) $\alpha(\boldsymbol{p},\boldsymbol{q})=\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}$ *and* $\varepsilon\boldsymbol{q}$ *is the projection* $\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}\to\hat{\boldsymbol{q}}$ *(*$q\ge 0$*)*, and
(iii) $\alpha(\boldsymbol{p},\boldsymbol{q})=\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}})$ (4.5) *and* $\varepsilon\boldsymbol{q}$ *is the composition*

$$\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}})\xrightarrow{\ \pi\ }\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}\xrightarrow{\text{proj.}}\hat{\boldsymbol{q}}.$$

Proof. Given a homotopy $h\colon \boldsymbol{X}\times\hat{\mathbf{1}}\to\boldsymbol{Y}\in\mathbf{RelCat}$, the desired homotopy in $\boldsymbol{S}$ is the composition

$$N_\alpha\boldsymbol{X}\times\Delta[0,1]\longrightarrow N_\alpha\boldsymbol{X}\times N_\alpha\hat{\mathbf{1}}\approx N_\alpha(\boldsymbol{X}\times\hat{\mathbf{1}})\xrightarrow{N_\alpha h}N_\alpha\boldsymbol{Y}$$

in which the isomorphism in the middle is due to the fact that N_α as a right adjoint preserves products, while the first map is induced by the composition

$$\Delta[0,1]\approx N_\iota\hat{\mathbf{1}}\longrightarrow N_\alpha\hat{\mathbf{1}},$$

in which the first map is the obvious isomorphism, while the second is induced by the natural transformation $\varepsilon\colon\alpha\to\iota$.

8. Proof of lemma 5.4

To prove lemma 5.4 we have to show that *for every object* $\boldsymbol{X}\in\mathbf{RelCat}$ *and integer* $p\ge 0$*, the map*

$$\pi_p^*\colon N\boldsymbol{X}(\boldsymbol{p},-)\longrightarrow N_\xi\boldsymbol{X}(\boldsymbol{p},-)\in\boldsymbol{S}$$

is a weak equivalence.

To prove this we recall that, for every pair of integers $p,q\ge 0$

$$N\boldsymbol{X}(\boldsymbol{p},\boldsymbol{q})=\mathbf{RelCat}(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}},\boldsymbol{X})\qquad\text{and}$$
$$N_\xi\boldsymbol{X}(\boldsymbol{p},\boldsymbol{q})=\mathbf{RelCat}\big(\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}),\boldsymbol{X}\big)$$

and embed the map π_p^* in a commutative diagram

$$\begin{tikzcd}
\mathbf{RelCat}(\check{\boldsymbol{p}}\times\hat{-},\boldsymbol{X}) \arrow[r, "a", "\approx"'] \arrow[d, "\pi_p^*"'] & \operatorname{diag}\bar{F}_p\boldsymbol{X} \arrow[r, "\operatorname{diag} f"] & \operatorname{diag}F_p\boldsymbol{X} \arrow[d, "\operatorname{diag} k"] \\
\mathbf{RelCat}\big(\xi(\check{\boldsymbol{p}}\times\hat{-}),\boldsymbol{X}\big) \arrow[r, "b", "\approx"'] & \operatorname{diag}\bar{G}_p\boldsymbol{X} \arrow[r, "\operatorname{diag} g"] & \operatorname{diag}G_p\boldsymbol{X}
\end{tikzcd}$$

in $\boldsymbol{S}$ and show that the maps a and b are isomorphisms and that the other three are weak equivalences.

The bisimplicial sets $\bar{F}_p\boldsymbol{X}$, $F_p\boldsymbol{X}$, $G_p\boldsymbol{X}$ and $\bar{G}_p\boldsymbol{X}$ and the maps between them are defined as follows:

$$\begin{tikzcd}
\bar{F}_p\boldsymbol{X}(\boldsymbol{q},\boldsymbol{r}) = \mathbf{RelCat}(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}, \boldsymbol{X}^{\hat{\mathbf{0}}}) \arrow[d, "f"'] & \\
F_p\boldsymbol{X}(\boldsymbol{q},\boldsymbol{r}) = \mathbf{RelCat}(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}, \boldsymbol{X}^{\hat{\boldsymbol{r}}}) \arrow[r, phantom, "\overset{7.1\text{(ii)a}}{\approx}"] & \mathbf{RelCat}(\hat{\boldsymbol{r}}, \boldsymbol{X}^{\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}}) \arrow[d, "k"] \\
G_p\boldsymbol{X}(\boldsymbol{q},\boldsymbol{r}) = \mathbf{RelCat}\big(\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}), \boldsymbol{X}^{\hat{\boldsymbol{r}}}\big) \arrow[r, phantom, "\overset{7.1\text{(ii)a}}{\approx}"] & \mathbf{RelCat}(\hat{\boldsymbol{r}}, \boldsymbol{X}^{\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}})}) \\
\bar{G}_p\boldsymbol{X}(\boldsymbol{q},\boldsymbol{r}) = \mathbf{RelCat}\big(\xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}), \boldsymbol{X}^{\hat{\mathbf{0}}}\big), \arrow[u, "g"] &
\end{tikzcd}$$

where f and g are induced by the unique maps $\hat{\boldsymbol{r}} \to \hat{\mathbf{0}}$ and k is induced by the map $\pi\colon \xi(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}) \to \check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}$ (4.5).

It then follows readily from 7.2, 7.3(iii) and 7.5 that the restrictions

$$f(-,\boldsymbol{r}), \quad g(-,\boldsymbol{r}), \quad \text{and} \quad k(\boldsymbol{q},-) \in \boldsymbol{S} \qquad (q, r \geq 0)$$

are homotopy equivalences and hence weak equivalences. Moreover, as any map of bisimplicial sets that induces weak equivalences between either their rows or their columns also induces a weak equivalence between their diagonals, it follows that

$$\operatorname{diag} f, \quad \operatorname{diag} g, \quad \text{and } \operatorname{diag} k$$

are all weak equivalences.

Finally, to complete the proof, one notes that there are obvious isomorphisms a and b which make the diagram commute.

9. Dwyer maps

In preparation for the proof of theorem 6.1 (in §10 below) we here

(i) note (in 9.1, 9.2 and 9.3) that Dwyer maps (3.5) are closed under retracts, pushouts and (possibly transfinite) compositions,

(ii) discuss (in 9.4 and 9.5) a way of producing Dwyer maps which explains why our main result involves the *two-fold* subdivision $\xi = \xi_t\xi_i$ (4.5) and not, as one might have expected from Thomason's original result [T1], the *iterated* functors ξ_t^2 and ξ_i^2, and

(iii) use these results to show that *every monomorphism*

$$L \longrightarrow M \in \mathrm{s}\boldsymbol{S} \qquad (5.2)$$

gives rise to a Dwyer map (5.3)

$$K_\xi L \longrightarrow K_\xi M \in \mathbf{RelCat}.$$

9.1. **Proposition.** *Every retract of a Dwyer map* (3.5) *is a Dwyer map.*

Proof. Let $\boldsymbol{A} \to \boldsymbol{B}$ be a Dwyer *inclusion* (3.5), and let

$$\begin{tikzcd}
\boldsymbol{A}' \arrow[r] \arrow[d] & \boldsymbol{A}'' \arrow[r] \arrow[d, "\bar{f}"] & \boldsymbol{B}' \arrow[d, "f"] \\
\boldsymbol{A} \arrow[r] \arrow[d] & Z\boldsymbol{A} \arrow[r] \arrow[d, "\bar{g}"] & \boldsymbol{B} \arrow[d, "g"] \\
\boldsymbol{A}' \arrow[r] & \boldsymbol{A}'' \arrow[r] & \boldsymbol{B}'
\end{tikzcd}$$

be a commutative diagram in which $gf = 1_{\boldsymbol{B}'}$, the horizontal maps are relative inclusions (3.1), and (r, s) is a strong deformation retraction (3.4) of $Z\boldsymbol{A}$ (3.5) onto $\boldsymbol{A}$. Then a straightforward calculation yields that $\boldsymbol{A}'$ is a sieve on $\boldsymbol{B}'$, that $\boldsymbol{A}'' = Z\boldsymbol{A}'$ and that the pair (r', s') where

$$r' = \bar{g}r\bar{f} \qquad \text{and} \qquad s' = \bar{g}s\bar{f}\colon \bar{g}r\bar{f} \longrightarrow \bar{g}\bar{f} = 1_{\boldsymbol{A}''}$$

is the desired strong deformation retraction of $\boldsymbol{A}'' = Z\boldsymbol{A}'$ onto $\boldsymbol{A}'$.

9.2. **Proposition.** *Let*

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "s"] \arrow[d, "i"'] & \boldsymbol{C} \arrow[d, "j"] \\
\boldsymbol{B} \arrow[r, "t"] & \boldsymbol{D}
\end{tikzcd}$$

be a pushout diagram in **RelCat** *in which the map* $i\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a Dwyer map* (3.5). *Then*

(i) *the map* $j\colon \boldsymbol{C} \to \boldsymbol{D}$ *is a Dwyer map in which* $Z\boldsymbol{C} = Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C}$, *and*
(ii) *if* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are relative posets, then so is* $\boldsymbol{D}$.

Moreover

(iii) *the map* $t\colon \boldsymbol{B} \to \boldsymbol{D}$ *restricts to isomorphisms*

$$X\boldsymbol{A} \approx X\boldsymbol{C} \qquad \text{and} \qquad X\boldsymbol{A} \cap Z\boldsymbol{A} \approx X\boldsymbol{C} \cap Z\boldsymbol{C}$$

where $X\boldsymbol{A} \subset \boldsymbol{B}$ *and* $X\boldsymbol{C} \subset \boldsymbol{D}$ *denote the full relative subcategories spanned by the objects which are not in the image of* $\boldsymbol{A}$ *or* $\boldsymbol{C}$.

Proof. Assuming that the map $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a *relative inclusion* (3.1) one shows that $\boldsymbol{C}$ is a sieve in $\boldsymbol{D}$ by noting that the characteristic relative functor (3.5) $\boldsymbol{B} \to \hat{\boldsymbol{1}}$ and the map $\boldsymbol{C} \to \hat{\boldsymbol{1}}$ which sends all of $\boldsymbol{C}$ to 0 yield a map $x\colon \boldsymbol{D} \to \hat{\boldsymbol{1}}$ such that $x^{-1}0 = \boldsymbol{C}$ and one shows in a similar manner that $Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C}$ is a cosieve in $\boldsymbol{D}$. Furthermore, the strong deformation retraction (r, s) of $Z\boldsymbol{A}$ onto $\boldsymbol{A}$ induces a strong deformation retraction (r', s') of $Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C}$ onto $\boldsymbol{C}$ given by

$$\begin{aligned}
r' &= r \amalg_{\boldsymbol{A}} \boldsymbol{C}\colon Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C} \longrightarrow \boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C} \quad = \boldsymbol{C} \\
s' &= s \amalg_{\boldsymbol{A}} \boldsymbol{C}\colon \quad r \amalg_{\boldsymbol{A}} \boldsymbol{C} \longrightarrow 1_{Z\boldsymbol{A}} \amalg_{\boldsymbol{A}} \boldsymbol{C} = 1_{Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C}}.
\end{aligned}$$

This, together with the fact that $Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C}$ is a cosieve in $\boldsymbol{D}$, readily implies that

$$Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \boldsymbol{C} = Z\boldsymbol{C}.$$

To prove (iii) one notes that (i) the relative inclusion

$$\hat{\boldsymbol{0}} = \boldsymbol{A} \amalg_{\boldsymbol{A}} \hat{\boldsymbol{0}} \longrightarrow \boldsymbol{B} \amalg_{\boldsymbol{A}} \hat{\boldsymbol{0}}$$

is a Dwyer map in which $Z\hat{\mathbf{0}} = Z\boldsymbol{A} \amalg_{\boldsymbol{A}} \hat{\mathbf{0}}$ is obtained from $X\boldsymbol{A} \cap Z\boldsymbol{A}$ by adding a single object 0 and, for every object $B \in X\boldsymbol{A} \cap Z\boldsymbol{A}$ a single weak equivalence $0 \to B$ and similarly $\boldsymbol{B} \amalg_{\boldsymbol{A}} \hat{\mathbf{0}}$ is obtained from $X\boldsymbol{A}$ by adding a single object 0 and, for every object $B \in X\boldsymbol{A} \cap Z\boldsymbol{A}$ a single weak equivalence $0 \to B$. Clearly $\boldsymbol{D} \amalg_{\boldsymbol{C}} \hat{\mathbf{0}}$ admits a similar description in terms of $X\boldsymbol{C}$ and $Z\boldsymbol{C}$ and the desired result now follows from the observation that the map $\boldsymbol{B} \to \boldsymbol{D}$ induces an isomorphism

$$\boldsymbol{B} \amalg_{\boldsymbol{A}} \hat{\mathbf{0}} \approx \boldsymbol{D} \amalg_{\boldsymbol{C}} \hat{\mathbf{0}} \ .$$

Finally, to prove (ii), we note that if two objects $E, F \in \boldsymbol{D}$ are both in $\boldsymbol{C}$ or else both in $X\boldsymbol{C}$, then there is at most one map between them as $\boldsymbol{C}$, and, in view of (iii), the relative categories $X\boldsymbol{C} \approx X\boldsymbol{A} \subset \boldsymbol{A}$ are both relative posets. It thus remains to consider the case that $E \in \boldsymbol{C}$ and $F \subset X\boldsymbol{C}$. In that case, there is no map $F \to E \in \boldsymbol{D}$ (because $\boldsymbol{C}$ is a sieve in $\boldsymbol{D}$), and if there is a map $g\colon E \to F \in \boldsymbol{D}$, then $F \in Z\boldsymbol{C}$ and $g = (s'F)(r'g)$; hence g is unique because $r'g\colon E \to r'F$ is in $\boldsymbol{C}$ and therefore unique.

9.3. **Proposition.** *Every (possibly transfinite) composition of Dwyer maps is a Dwyer map.*

Proof. Assuming that all Dwyer maps involved are *relative inclusions* this follows readily from the following observations.

(i) If $\boldsymbol{A}_0 \to \boldsymbol{A}_1$ and $\boldsymbol{A}_1 \to \boldsymbol{A}_2$ are Dwyer maps with strong deformation retractions (3.4),

$$\begin{array}{ccc} (r_{0,1}, s_{0,1}) & \text{and} & (r_{1,2}, s_{1,2}) \\ \text{of } Z(\boldsymbol{A}_0, \boldsymbol{A}_1) \text{ onto } \boldsymbol{A}_0 & & \text{of } Z(\boldsymbol{A}_1, \boldsymbol{A}_2) \text{ onto } \boldsymbol{A}_1, \end{array}$$

then $\boldsymbol{A}_0$ is a sieve in $\boldsymbol{A}_2$, and one can obtain a strong deformation retraction

$$(r_{0,2}, s_{0,2}) \qquad \text{of } Z(\boldsymbol{A}_0, \boldsymbol{A}_2) \text{ onto } \boldsymbol{A}_0$$

that restricts on $Z(\boldsymbol{A}_0, \boldsymbol{A}_1)$ to $(r_{0,1}, s_{0,1})$ by "composing" the restriction $(r'_{1,2}, s'_{1,2})$ of $(r_{1,2}, s_{1,2})$ to $Z(\boldsymbol{A}_0, \boldsymbol{A}_1)$ with $(r_{0,1}, s_{0,1})$, i.e., by defining $(r_{0,2}, s_{0,2})$ by

$$r_{0,2} = r_{0,1} r'_{1,2} \qquad \text{and} \qquad s_{0,2} = s'_{1,2} s_{0,1}.$$

(ii) If λ is a limit ordinal, and

$$\boldsymbol{A}_0 \longrightarrow \cdots \longrightarrow \boldsymbol{A}_\beta \longrightarrow \qquad (\beta \le \lambda)$$

is a sequence of relative inclusions such that
   (a) for every limit ordinal $\gamma \le \lambda$,one has $\boldsymbol{A}_\gamma = \bigcup_{\alpha<\gamma} \boldsymbol{A}_\alpha$,
   (b) for all $\beta < \lambda$, $\boldsymbol{A}_0$ is a sieve in $\boldsymbol{A}_\beta$, and
   (c) there exist strong deformation retractions

$$(r_{0,\beta}, s_{0,\beta}) \qquad \text{of } Z(\boldsymbol{A}_0, \boldsymbol{A}_\beta) \text{ onto } \boldsymbol{A}_0$$

   (one for each $\beta < \lambda$) such that, for each $\alpha < \beta < \lambda$, $(r_{0,\alpha}, s_{0,\alpha})$ is the restriction of $(r_{0,\beta}, s_{0,\beta})$ to $Z(\boldsymbol{A}_0, \boldsymbol{A}_\alpha)$,

then $\boldsymbol{A}_0$ is a sieve in $\boldsymbol{A}_\lambda$ and there exists a (unique) strong deformation retraction $(r_{0,\lambda}, s_{0,\lambda})$ of $Z(\boldsymbol{A}_0, \boldsymbol{A}_\lambda)$ onto $\boldsymbol{A}_0$ such that, for every $\beta < \lambda$, $(r_{0,\beta}, s_{0,\beta})$ is the restriction of $(r_{0,\lambda}, s_{0,\lambda})$ to $Z(\boldsymbol{A}_0, \boldsymbol{A}_\beta)$.

9.4. **Proposition.** If

(i) $\boldsymbol{P} \to \boldsymbol{Q} \in \mathbf{RelPos}$ *is a relative inclusion* (3.1), *and*
(ii) $\boldsymbol{P}$ *is a cosieve in* $\boldsymbol{Q}$ (3.5),

then the induced inclusion $\xi_t\boldsymbol{P} \to \xi_t\boldsymbol{Q}$ (4.1) *is a Dwyer map* (3.5).

Proof. For every object $(a_0, \ldots, a_n \in \xi_t\boldsymbol{Q}$ (4.5(iii)) either

(i) none of the a_i $(0 \le i \le n)$ is in $\boldsymbol{P}$, or
(ii) there is (in view of 9.4(ii)) an integer j with $0 \le j \le n$ such that $a_j \in \boldsymbol{P}$ iff $i \ge j$, in which case
(iii) $(a_0, \ldots, a_n) \in \xi_t\boldsymbol{P}$ and $(a_0, \ldots, a_n) \in Z\xi_t\boldsymbol{P}$.

It now readily follows that $\xi_t\boldsymbol{P}$ is a sieve in $\xi_t\boldsymbol{Q}$ and that the strong deformation retraction (r,s) given by the formulas

$$r(a_0, \ldots, a_n) = (a_j, \ldots, a_n) \in \xi_t\boldsymbol{P}$$
$$s(a_0, \ldots, a_n) = (a_j, \ldots, a_n) \longrightarrow (a_0, \ldots, a_n) \in \xi_t\boldsymbol{Q}$$

is the desired one.

9.5. **Proposition.** *For every pair of integers* $p, q \ge 0$ *let* $\partial\Delta[p,q] \subset \Delta[p,q] \in \mathrm{s}\boldsymbol{S}$ (5.2) *denote the largest subobject not containing its (only) non-degenerate (in both directions)* (p,q)*-bisimplex. Then the inclusion* $\partial\Delta[p,q] \to \Delta[p,q]$ *induces* (5.3) *a Dwyer map*

$$K_\xi\partial\Delta[p,q] \longrightarrow K_\xi\Delta[p,q] = \xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) = \xi_t\xi_i(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \in \mathbf{RelPos} \qquad (4.1)$$

Proof. Let $K_{\xi_i}\colon \mathrm{s}\boldsymbol{S} \to \mathbf{RelCat}$ denote the colimit preserving functor which, for every pair of integers $a, b \ge 0$, sends $\Delta[a,b]$ to $\xi_i(\check{\boldsymbol{a}} \times \hat{\boldsymbol{b}})$. We show that

I the inclusion $\partial\Delta[p,q] \to \Delta[p,q]$ induces an inclusion
$$K_{\xi_i}\partial\Delta[p,q] \to K_{\xi_i}\Delta[p,q]$$
that satisfies 9.4(i) and 9.4(ii), implying that the resulting inclusion
$$\xi_tK_{\xi_i}\partial\Delta[p,q] \to \xi_tK_{\xi_i}\Delta[p,q] = K_\xi\Delta[p,q]$$
is a Dwyer inclusion, and
II $K_\xi\partial\Delta[p,q] = \xi_tK_{\xi_i}\partial\Delta[p,q]$.

To show these, let $\boldsymbol{D}$ denote the poset that has as its objects the subcategories of $\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}$ of the form $\check{\boldsymbol{a}} \times \hat{\boldsymbol{b}}$ for which $\check{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ are relative subcategories of $\check{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$, respectively, and as its morphisms the relative inclusions. One readily verifies the following.

(i) For every pair of objects $\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1$ and $\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2 \in \boldsymbol{D}$ for which both $\check{\boldsymbol{a}}_1 \cap \check{\boldsymbol{a}}_2$ and $\hat{\boldsymbol{b}}_1 \cap \hat{\boldsymbol{b}}_2$ are nonempty,
  (a) $(\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1) \cap (\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2) = (\check{\boldsymbol{a}}_1 \cap \check{\boldsymbol{a}}_2) \times (\hat{\boldsymbol{b}}_1) \cap \hat{\boldsymbol{b}}_2)$
  (b) $\xi_i(\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1) \cap \xi_i(\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2) = \xi_i((\check{\boldsymbol{a}}_1 \cap \check{\boldsymbol{a}}_2) \times (\hat{\boldsymbol{b}}_1) \cap \hat{\boldsymbol{b}}_2))$
  (c) $\xi(\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1) \cap \xi(\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2) = \xi((\check{\boldsymbol{a}}_1 \cap \check{\boldsymbol{a}}_2) \times (\hat{\boldsymbol{b}}_1) \cap \hat{\boldsymbol{b}}_2))$.
(ii) For every map $f\colon \check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1 \to \check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2 \in \boldsymbol{D}$,
  (a) ξ_if is a relative inclusion, and $\xi_i(\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1)$ is a cosieve in $\xi(\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2)$, and
  (b) ξf is a relative inclusion, and $\xi(\check{\boldsymbol{a}}_1 \times \hat{\boldsymbol{b}}_1)$ is a sieve in $\xi(\check{\boldsymbol{a}}_2 \times \hat{\boldsymbol{b}}_2)$.

One verifies I above by noting, in view of (i)b and (ii)a, that $K_{\xi_i}\partial\Delta[p,q]$ is exactly the union in $\xi_i(\check{\boldsymbol{p}}\times\hat{\boldsymbol{q}}) = K_{\xi_i}\Delta[p,q]$ of all the $\xi_i(\check{\boldsymbol{a}}\times\hat{\boldsymbol{b}})$'s, and thus the resulting inclusion $K_{\xi_i}\partial\Delta[p,q]\to K_{\xi_i}\Delta[p,q]$ satisfies 9.4(i) and 9.4(ii).

To obtain II above one first notes that, as above, the map $K_\xi\partial\Delta[p,q]\to K_\xi\Delta[p,q]$ is an inclusion, and thus the obvious map $K_\xi\partial\Delta[p,q]\to\xi_t K_{\xi_i}\partial\Delta[p,q]$ is also an inclusion. It remains therefore to show that this map is onto. But this follows from the fact that, for every map $h\colon x\to y\in\xi_t K_{\xi_i}\partial\Delta[p,q]$, where y is a monomorphism $\boldsymbol{n}\to K_{\xi_i}\partial\Delta[p,q]$, the object $y0\in K_{\xi_i}\partial\Delta[p,q]$ lies in some $\xi_i(\check{\boldsymbol{a}}\times\hat{\boldsymbol{b}})$ and hence, in view of (ii)b, the whole map $h\colon x\to y$ lies in $K_\xi\partial\Delta[p,q]$.

Finally we show, by combining 9.5 with 9.2 and 9.3,

9.6. **Proposition.** Every monomorphism $L\to M\in \mathrm{s}\boldsymbol{S}$ induces (5.3) a Dwyer map

$$K_\xi L\longrightarrow K_\xi M\in\mathbf{RelPos}.$$

Proof. Assume that L is actually a subobject of $\boldsymbol{M}$ and denote by M^n $(n\geq -1)$ the smallest subobject containing all (i,j)-bisimplices with $i+j\leq n$. Then

(i) $M=\bigcup_{n\geq -1}(M^n\cup L)$ and $K_\xi M=\bigcup_{n\geq -1}K_\xi(M^n\cup L)$.

Furthermore if $\Delta_n(M,L)$ (resp. $\partial\Delta_n(M,L)$) $(n\geq 0)$ denotes the disjoint union of copies of $\Delta[i,j]$ (resp. $\partial\Delta[i,j]$), one for each non-degenerate (in both directions) (i,j)-bisimplex with $i+j=n$ that is in $M^n\cup L$, but not in $M^{n-1}\cup L$, then 9.2(i) and 9.5 imply:

(ii) The pushout diagram in $\mathrm{s}\boldsymbol{S}$

$$\begin{tikzcd}
\partial\Delta_n(M,L) \arrow[r] \arrow[d] & M^{n-1}\cup L \arrow[d] \\
\Delta_n(M,L) \arrow[r] & M^n\cup L
\end{tikzcd}$$

induces a pushout diagram in **RelCat**

$$\begin{tikzcd}
K_\xi\partial\Delta_n(M,L) \arrow[r] \arrow[d] & K_\xi(M^{n-1}\cup L) \arrow[d] \\
K_\xi\Delta_n(M,L) \arrow[r] & K_\xi(M^n\cup L),
\end{tikzcd}$$

in which the vertical maps are Dwyer maps. It therefore follows from (i) and 9.3 that the map $K_\xi L\to K_\xi M$ is a Dwyer map as well.

That this map is in **RelPos**, i.e. that $K_\xi M$ (and hence $K_\xi L$) is a relative poset now can be shown by combining the above for $L=\emptyset$ with 9.2(ii) and the fact that every (possibly transfinite) composition of relative inclusions of relative posets is again a relative inclusion of relative posets.

10. Proof of theorem 6.1

Before proving theorem 6.1 (in 10.5 below) we

(i) obtain a key lemma which states that, up to a weak equivalence in the Reedy model structure on $\mathrm{s}\boldsymbol{S}$ (and hence in any left Bousfield localization thereof), pushing out along a Dwyer map commutes with applying the (simplicial) nerve functor $N\colon \mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$ (and hence (5.4) also the functor $N_\xi\colon \mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$),

and then

(ii) use this to show that the unit $1 \to N_\xi K_\xi$ of the adjunction

$$K_\xi\colon \mathrm{s}\boldsymbol{S} \longleftrightarrow \mathbf{RelCat} : N_\xi$$

is a natural Reedy weak equivalence, which in turn readily implies that a map $f\colon L \to M \in \mathrm{s}\boldsymbol{S}$ is a weak equivalence in the Reedy model structure or any left Bousfield localization thereof iff the induced map $N_\xi K_\xi f\colon N_\xi K_\xi L \to N_\xi K_\xi M \in \mathrm{s}\boldsymbol{S}$ is so.

We thus start with

10.1. **Another key lemma.** *Let*

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "s"] \arrow[d, "i"'] & \boldsymbol{C} \arrow[d, "j"] \\ \boldsymbol{B} \arrow[r, "t"] & \boldsymbol{D} \end{tikzcd}$$

be a pushout diagram in $\mathbf{RelCat}$ *in which the map* $i\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a Dwyer map* (3.5)*. Then, in the Reedy model structure on* $\mathrm{s}\boldsymbol{S}$ *(and hence in any left Bousfield localization thereof),*

(i) *the induced map*

$$N\boldsymbol{B} \amalg_{N\boldsymbol{A}} N\boldsymbol{C} \longrightarrow N\boldsymbol{D} \in \mathrm{s}\boldsymbol{S}$$

is a weak equivalence, and

(ii) *if Ni is a weak equivalence, then so is Nj and if Ns is a weak equivalence, then so is Nt.*

Proof. One readily verifies that (3.5 and 9.2(iii))

$$X\boldsymbol{A},\ Z\boldsymbol{A} \text{ and } X\boldsymbol{A} \cap Z\boldsymbol{A}$$

are cosieves in $\boldsymbol{B}$ and that therefore the image of a map $\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}} \to \boldsymbol{B}$ $(p, q \geq 0)$ is

(i) either only in $X\boldsymbol{A}$,
(ii) or only in $Z\boldsymbol{A}$
(iii) or both in $X\boldsymbol{A}$ and in $Z\boldsymbol{A}$

iff the image of the initial object $(0,0) \in \check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}$ is. It follows that $N\boldsymbol{B}$ and (9.2(i)) $N\boldsymbol{D}$ are pushouts

$$N\boldsymbol{B} \overset{b}{\approx} NX\boldsymbol{A} \amalg_{N(X\boldsymbol{A}\cap Z\boldsymbol{A})} NZ\boldsymbol{A} \qquad \text{and}$$

$$N\boldsymbol{D} \overset{d}{\approx} NX\boldsymbol{C} \amalg_{N(X\boldsymbol{C}\cap Z\boldsymbol{C})} NZ\boldsymbol{C}$$

and that therefore the map $N\boldsymbol{B} \amalg_{N\boldsymbol{A}} N\boldsymbol{C} \to N\boldsymbol{D}$ admits a factorization

$$\begin{aligned} N\boldsymbol{B} \amalg_{N\boldsymbol{A}} N\boldsymbol{C} \xrightarrow{a} N\boldsymbol{B} \amalg_{NZ\boldsymbol{A}} NZ\boldsymbol{C} &\overset{b}{\approx} \\ NX\boldsymbol{A} \amalg_{N(X\boldsymbol{A} \cap Z\boldsymbol{A})} NZ\boldsymbol{A} \amalg_{NZ\boldsymbol{A}} NZ\boldsymbol{C} &= \\ NX\boldsymbol{A} \amalg_{N(X\boldsymbol{A} \cap Z\boldsymbol{A})} NZ\boldsymbol{C} \overset{c}{\approx} NX\boldsymbol{C} \amalg_{N(X\boldsymbol{C} \cap Z\boldsymbol{C})} NZ\boldsymbol{C} &\overset{d}{\approx} N\boldsymbol{D} \end{aligned}$$

in which c is induced by the isomorphisms of 9.2(iii), and a is induced by the inclusions $\boldsymbol{A} \to Z\boldsymbol{A}$ and $C \to Z\boldsymbol{C}$ (3.5).

Part (i) now follows from the observation that, in view of 7.5(ii) and the fact that (3.3 and 3.5) the maps $\boldsymbol{A} \to Z\boldsymbol{A}$ and $\boldsymbol{C} \to Z\boldsymbol{C}$ are homotopy equivalences, the induced maps

$$N\boldsymbol{A} \longrightarrow NZ\boldsymbol{A} \qquad \text{and} \qquad N\boldsymbol{C} \longrightarrow NZ\boldsymbol{C} \in \mathrm{s}\boldsymbol{S}$$

are weak equivalences.

Furthermore the first half of (ii) is an immediate consequence of (i), while the second half follows from (i) and the left properness of the model structures involved.

10.2. **Corollary.** In view of lemma 5.4 proposition 10.1 remains valid if one replaces everywhere the functor N by N_ξ (5.3).

10.3. **Proposition.** *The unit*

$$\eta_\xi \colon 1 \longrightarrow N_\xi K_\xi$$

of the adjunction $K_\xi \colon \mathrm{s}\boldsymbol{S} \leftrightarrow \mathbf{RelCat} : N_\xi$ (5.3) is a natural weak equivalence in the Reedy model structure on $\mathrm{s}\boldsymbol{S}$ (and hence also any left Bousfield localization thereof).

10.4. **Corollary.** *A map $f \colon L \to M \in \mathrm{s}\boldsymbol{S}$ is a weak equivalence in the Reedy model structure or any of its left Bousfield localizations iff the induced map $N_\xi K_\xi L \to N_\xi K_\xi M \in \mathrm{s}\boldsymbol{S}$ is so.*

Proof of 10.3. We first show that

($*$) *for every pair of integers $p, q \geq 0$, the map*

$$\eta_\xi \colon \Delta[p,q] \longrightarrow N_\xi K_\xi \Delta[p,q] \in \mathrm{s}\boldsymbol{S}$$

is a weak equivalence.

This follows from the observation that, in the commutative diagram

$$\begin{array}{ccc} \Delta[p,q] & \xrightarrow{\eta_\xi} & N_\xi K_\xi \Delta[p,q] = N_\xi \xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}) \\ \big\downarrow & & \big\downarrow \pi_* \\ NK\Delta[p,q] & \xrightarrow{\pi^*} & N_\xi K \Delta[p,q] = N_\xi \xi(\check{\boldsymbol{p}} \times \hat{\boldsymbol{q}}), \end{array}$$

in which η denotes the unit of the adjunction $K \colon \mathrm{s}\boldsymbol{S} \leftrightarrow \mathbf{RelCat} : N$ (5.3) and π is as in 4.5(i). η is readily verified to be a Reedy equivalence, while π^* and π_* are so in view of 5.4 and 7.5(iii) and 7.3(iii) respectively.

To deal with an arbitrary object $M \in \mathrm{s}\boldsymbol{S}$ one notes that, in the notation of the proof of 9.6,

$$M = \bigcup_n M^n \qquad \text{and} \qquad N_\xi K_\xi M = \bigcup_n N_\xi K_\xi M^n,$$

and that it thus suffices to prove that

$(*)_n$ *for every integer $n \geq 0$, the map*

$$\eta_\xi \colon M^n \longrightarrow N_\xi K_\xi M^n \in \mathrm{s}\boldsymbol{S}$$

is a weak equivalence.

For $n = 0$ this is obvious, and we thus show that, for $n > 0$, $(*)_{n-1}$ implies $(*)_n$.

To do this, consider the commutative diagram, in which $\Delta_n(M, \emptyset)$ and $\partial\Delta_n(M, \emptyset)$ are as in the proof of 9.6,

$$\begin{tikzcd}
\partial\Delta_n(M,\emptyset) \arrow[rr] \arrow[dr] \arrow[dd] & & \Delta_n(M,\emptyset) \arrow[dr] \arrow[dd] & & \\
& N_\xi K_\xi \partial\Delta_n(M,\emptyset) \arrow[rr] \arrow[dd] & & N_\xi K_\xi \Delta_n(M,\emptyset) \arrow[dd] \arrow[ddr, bend left] & \\
M^{n-1} \arrow[rr] \arrow[dr] & & M^n \arrow[dr] & & \\
& N_\xi K_\xi M^{n-1} \arrow[rr] \arrow[drrr, bend right] & & H \arrow[dr] & \\
& & & & N_\xi K_\xi M^n
\end{tikzcd}$$

in which the two squares are pushout squares and all maps are the obvious ones. It then follows from $(*)$ and $(*)_{n-1}$ above that the slanted maps at the left and the top are weak equivalences and so is therefore the map $M^n \to H$. The desired result now follows from the observation that, in view of 9.6, 10.1, and 10.2 so is the map $H \to N_\xi K_\xi M^n$.

Now we are finally ready for the

10.5. **Proof of theorem 6.1.**

(i) *The model structure.* To show that the Reedy model structure on $\mathrm{s}\boldsymbol{S}$ lifts to a model structure on **RelCat** one has to verify 5.1(i) and 5.1(ii). Clearly 5.1(i) follows from the smallness of the prospective generating cofibrations and generating trivial cofibrations. To show that 5.1(ii) holds, one notes that, in view of 10.4, the right adjoint N_ξ sends every prospective generating trivial cofibration to a weak equivalence in $\mathrm{s}\boldsymbol{S}$ and that, in view of 9.2, 9.3, 9.6, 10.1 and 10.2, the same holds for every (possibly transfinite) composition of pushouts of the prospective generating trivial cofibrations. Moreover, in view of [H, Th. 3.3.20], all this applies also to any Bousfield localization of the Reedy structure.

Furthermore

(a) 6.1(i) and 6.1(ii) follow from 5.1(iv),

(b) 6.1(iii) follows from 9.1, 9.2, 9.3 and 9.6, and

(c) 6.1(iv) follows similarly from 9.1, 9.2, 9.3 and the fact that the colimit of every (possibly transfinite) sequence of monomorphisms of posets is again a poset.

(ii) *The Quillen equivalence.* This follows readily from 6.1(i) and 10.3.

And finally

(iii) *The (left) properness.* Left properness follows from 10.1(ii), and 10.3 and the left properness of the model structures on $\mathrm{s}\boldsymbol{S}$. The right properness of the model structure lifted from the Reedy model structure is a consequence

of the right properness of the latter and the fact that the right adjoint preserves limits.

References

[BK1] C. Barwick and D. M. Kan, *A characterization of simplicial localization functors*. To appear.

[BK2] ———, *In the category of relative categories the Rezk equivalences are exactly the DK equivalences*. To appear.

[C] D.-C. Cisinski, *La classe des morphismes de Dwyer n'est pas stable par rétractes*, Cahiers Topologie Géom. Différentielle Catég. **40** (1999), no. 3, 227–231.

[DK] W. G. Dwyer and D. M. Kan, *Calculating Simplicial Localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.

[H] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys and Monographs, vol. 99, AMS, 2003.

[K] D. M. Kan, *On c. s. s. complexes*, Amer. J. Math. **79** (1957), 449–476.

[LTW] D. M. Latch, R. W. Thomason, and W. S. Wilson, *Simplicial sets from categories*, Math. Z. **164** (1979), 195–214.

[M] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[R] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[T1] R. W. Thomason, **Cat** *as a closed model category*, Cahiers de Topologie et Géometrie Differentielle **21** (1980), no. 3, 305–324.

[T2] B. Toën, *Vers une axiomatisation de la théorie des catégories supérieures*, K-Theory **34** (2005), no. 3, 233–263.

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